Definition (positively homogeneous sub-additive)

A real valued function $p: X \to \mathbb{R}$ on a vector space X is called positively homogeneous sub-additive if

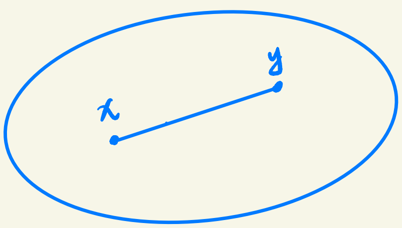
(i) $p(\alpha x) = \alpha p(x)$ for all $x \in X$ and $\alpha > 0$;

(ii) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$

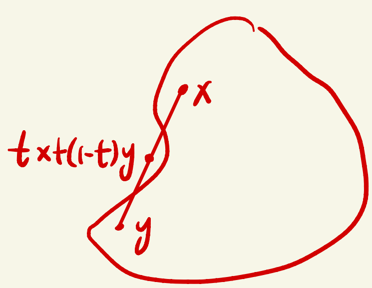
Rmk: A norm is a positively homogeneous sub-additive function.

Definition (Convex set)

A subset D of a vector space X is called convex if $tx + (1-t)y \in D$ for all $x, y \in D$ and $t \in (0,1)$.



$x, y \in D \Rightarrow$ the segment connecting $x, y \subset D$



non-convex

Definition (Minkowski functional)

Let X be a normed space and $D \subset X$ a convex subset. Suppose $0 \in D^\circ$. Define $\mu := \mu_D : X \to [0, \infty)$ by

$$\mu(x) = \inf\{t > 0 : x \in tD\}.$$

Rmk: (i) μ is well-defined, i.e., for any $x \in X$, $\exists\, t > 0$ such that $x \in tD$. This is due to $0 \in D^\circ$.

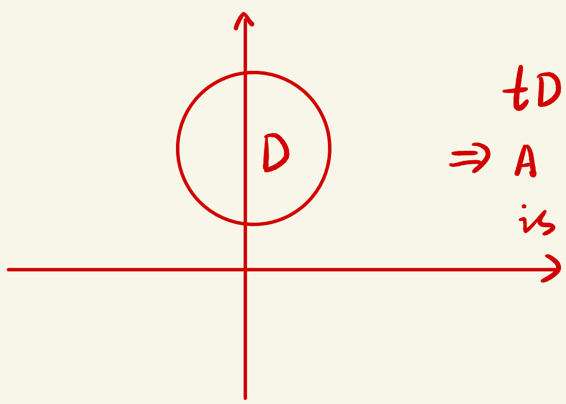
Pf: Since $0 \in D^\circ$, $\exists\, \delta > 0$ such that $B_\delta(0) \subset D$.

Thus $B_{t\delta}(0) = tB_\delta(0) \subset tD$.

For any $x \in X$, take $t > 0$ such that $t\delta > \|x\|$.

Hence, $x \in B_{t\delta}(0) \subset tD$.

(ii) If $0 \notin D^\circ$, $\{t > 0 : x \in tD\}$ may be empty.



tD is always on the upper half plane

$\Rightarrow$ A point on the lower half plane is not in tD for any $t > 0$.

Proposition

For any convex subset D of X whose interior containing 0, $\mu := \mu_D$ is a positive homogeneous sub-additive function.

Proof: (i) μ is positive homogeneous.

Pick any $x \in X$ and $\alpha > 0$.

$$
\begin{aligned}
\mu(\alpha x) &= \inf\{t>0 : \alpha x \in tD\} \\
&= \inf\{t>0 : x \in \tfrac{t}{\alpha} D\} \\
&= \alpha \inf\{\tfrac{t}{\alpha} > 0 : x \in \tfrac{t}{\alpha} D\} \\
&= \alpha \inf\{s>0 : x \in sD\} \\
&= \alpha \mu(X)
\end{aligned}
$$

(ii) μ is sub-additive.

Pick any $x, y \in X$. Fix $\varepsilon > 0$.

By definition, there exists $s_1 < \mu(x) + \varepsilon$ and $s_2 < \mu(y) + \varepsilon$ such that $x \in s_1 D$ and $y \in s_2 D$, i.e., $\frac{x}{s_1} \in D$ and $\frac{y}{s_2} \in D$.

We want to find $t \in (0,1)$ such that

$t \frac{x}{s_1} + (1-t)\frac{x}{s_2} = \alpha(x+y)$, i.e., $\frac{t}{s_1} = \frac{1-t}{s_2}$.

Take $t = \frac{s_1}{s_1+s_2}$. Since D is convex and $\frac{x}{s_1}, \frac{y}{s_2} \in D$,

then $t\frac{x}{s_1} + (1-t)\frac{y}{s_2} \in D$, i.e., $\frac{x}{s_1+s_2} + \frac{y}{s_1+s_2} \in D$.

This is equivalent to $x+y \in (s_1+s_2)D$.

By definition,

$$\begin{aligned} \mu(x+y) &= \inf\{t>0 : x+y \in tD\} \\ &\le s_1 + s_2 \\ &< \mu(x)+\varepsilon+\mu(y)+\varepsilon \\ &= \mu(x)+\mu(y)+2\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ gives $\mu(x+y) \le \mu(x)+\mu(y)$.

$\square$